Master thesis

## A mathematical model for the influence of the social insensitivity on the SIS epidemic dynamics

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#### Ying Xie

#### B9IM1019

Department of Computer and Mathematical Sciences Graduate School of Information Sciences Tohoku University Aramaki-Aza-Aoba 6-3-09, Aoba-ku, Sendai, Miyagi 980-8579 JAPAN

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- **G** Oscillatory behavior in the case of B = 0 33
- **H** Oscillatory behavior in the case of a = 0 38

# Chapter 1 Introduction

The last two decades have witnessed a number of major outbreaks of transmissible disease including SARA, Ebola, and most recently, the coronavirus. Globally, as of 10 November 2020, there have been over 50 millions confirmed cases of COVID-19, including over 1,257 thousands deaths (WHO, 2020). In many researchers' work, the individual behavior is considered as a key to understand the epidemiological system. When the spread of a transmissible disease is recognized in a community, various media (e.g., TV, newspaper, social networking sites) (Kiss et al. 2010) can send such messages as to alert to the members the presence of a disease spreading over the community. Then, the community induces some behaviors of its members, for instance, waring mask, limiting the number of contacts with others, taking medication, vaccination. They include qualitative or quantitative changes in the quotidian behavior. Such behavioral changes may reduce the susceptibility to the disease or increase the recovery rate from it. Generally, the report by media on a large number of infected individuals is more likely to alert individuals to take some preventative behaviors. In this paper, we call the collective effect of such individual behaviors on the dynamics of disease spread by *social response*. However, the community may not respond to a transmissible disease even though such a disease is spreading in the community.

Funk *et al.* (2010) quantified the impact on the endemicity of a disease in a wellmixed population under the variation of different disease parameters as a consequence of growing awareness in the population. Agaba *et al.* (2017) investigated how the dissemination of private awareness arising from direct contacts between unaware and aware individuals and that of public awareness stem withpopulation-wide campaigns affect the dynamics of the disease spread. Both works focused on the substantial fraction of the population, though the effect of social response was not taken into account. Misra *et al.* (2011) proposed a non-linear mathematical model to discuss the effect of awareness about the spread of infectious diseases. They considered the awareness to induce the isolation from the contact to others. In contrast to Misra *et al.* (2011), Basir *et al.* (2018) assumed that the rate of becoming aware (resp; unaware) is related to the media campaign.

In this paper, a mathematical model is proposed to consider the influence of the

social insensitivity on the epidemic dynamics of transmissible disease, introducing the infection rate and the recover rate which are affected by the social response to the disease spread.

In Chapter 2 of this paper, we construct the generic model with the influence of the social response, and investigate some general nature of it in Chapter 3. In Chapter 4, we introduce specific functions for the coefficient of the infection rate and the recovery rate with respect to the social response, and analyze it in detail, focusing on the effect of social insensitivity on the nature of disease spread.

### Chapter 2

## SIS modeling with social response

#### 2.1 Assumptions

When a transmissible disease spreads in a community, changes in behavior in response to the outbreak can alter the progression of its spread. To construct a mathematical model, we set up now the following assumptions about the epidemic dynamics and the social response:

- The disease is infatal and the disease-induced death can be negligible (for example, the common cold).
- Such *social insensitivity* could be caused by the weak influence of the corresponding alert or by the unconcern to the disease spread, which are affected by the education, the culture, and the history of the community.
- The recovered individual can not get effectively long-lasting immunity and becomes susceptible again in a certain period after the recovery.
- The demographic change of the community is negligible in the time scale of considered epidemic dynamics.
- The effect of social response appears as the reduction of infection rate and the increase of recovery rate. For example, the social response may result in a decrease of individual contacts.
- The social response has a decay rate in time, while the existence of infectives in the community may enhance it.
- The disease spread may not enhance the social response if the number of infectives is small enough to make the people unconcern to it, that is, to cause the social insensitivity.

If the community makes stronger social response to the disease, such as reducing the social contacts, the transmission rate may become smaller and the recovery rate may

become larger. On the other side, the community may show a negative attitude or take no response to such disease due to the social insensitivity.

#### 2.2 Generic Model

Let S(t) and I(t) be the susceptible and the infective densities at time t, and consider the following model:

$$\frac{dS}{dt} = -\beta(M)SI + q(M)I$$

$$\frac{dI}{dt} = \beta(M)SI - q(M)I$$

$$\frac{dM}{dt} = \Gamma(I) - \mu M,$$
(2.1)

where M = M(t) is the strength of the social response at time t, and  $\mu$  is its fading rate. Let set up the initial condition that S(0) > 0, I(0) > 0 and M(0) = 0, which means that there is no social response at t = 0. At the beginning of the disease spread, people know little information about it and unconcern to it. The coefficient of the disease transmission rate  $\beta = \beta(M)$  is given by a continuous and decreasing function of M, while the recover rate q = q(M) is given by a continuous and increasing function of M.

Let us call  $\Gamma(I)$  by the social sensitivity function, which represents the nature of social response including the sensitivity to the disease spread. In this paper, we assume that the community takes no response as long as the number of infectives is below a critical value  $I_c$ :

$$\Gamma(I) := \begin{cases} 0 & \text{for } I \leq I_c; \\ \gamma(I - I_c) & \text{for } I > I_c. \end{cases}$$
(2.2)

 $\gamma$  is corresponding to the social sensitivity, while  $I_c$  is the threshold value for the number of infectives to cause the social response. When the number of infectives is sufficient small, there is no awareness nor information about the disease to the public. As the number of infectives becomes larger, the information about a transmissible disease may become more frequent.

Since S(t) + I(t) = N(t) with a constant N for any  $y \ge 0$ , and  $N(t) \equiv N > 0$ , the above system (2.1) can be reduced to the following 2-dimensional one:

$$\frac{dI}{dt} = \beta(M)(N-I)I - q(M)I$$

$$\frac{dM}{dt} = \Gamma(I) - \mu M.$$
(2.3)

Note that I(t) < N for any  $t \ge 0$ .

### Chapter 3

## Analysis on the generic model

#### 3.1 Basic reproduction number

The basic reproduction number  $\mathscr{R}_0$ , which is defined as the expected number of secondary infectives who is infected, in a totally susceptible community, by a single infected individual during the time span of the infection.  $\mathscr{R}_0$  is generally used to measure the transmission potential of a disease. To derive  $\mathscr{R}_0$  for model (2.1), we consider the condition such that dI/dt > 0 for  $I(0) \ll 1$  and  $S(0) \approx N$ . Then, we can easily obtain the following condition with definition of  $\mathscr{R}_0$  for (2.1):

$$\mathscr{R}_0 := \frac{\beta_0 N}{q_0} > 1, \tag{3.1}$$

when the number of infectives increases in an early period after the invasion of the disease. If  $\mathscr{R}_0 < 1$ , the number of infectives monotonically decreases after the invasion.

#### 3.2 Model with no social response

If there is no social response, then M(t) = 0 for any  $t \ge 0$ . Thus, we have

$$\frac{dI}{dt} = \beta_0 (N - I)I - q_0 I \tag{3.2}$$

with  $\beta(0) = \beta_0 > 0$  and  $q(0) = q_0 > 0$ . This can be rewritten to the following with the basic reproduction number  $\mathscr{R}_0$ :

$$\frac{dI}{dt} = \beta_0 N (1 - 1/\mathscr{R}_0) \left\{ 1 - \frac{I}{N(1 - 1/\mathscr{R}_0)} \right\} I.$$
(3.3)

The equation (3.2) or (3.3) can be regarded as Verhulst model, from the name of P. Verhulst who, some decades after Malthus, participated in the discussion about the population explosion, proposing this model (Verhulst, 1838). Today it is well-known as



Figure 3.1: Numerical calculation of the temporal variation of I(t) by (3.3). For three different initial values of I(0) with  $\mathscr{R}_0 = 2.0$ ,  $\beta_0 = 1.0$ , N = 1000.0.

the logistic equation. Model (3.3) with initial condition  $I(0) = I_0 > 0$  can be easily solved:

$$I(t) = N(1 - 1/\mathscr{R}_0) \frac{I_0}{I_0 + \{N(1 - 1/\mathscr{R}_0) - I_0\}e^{-\beta_0 N(1 - 1/\mathscr{R}_0)t}}$$

When  $\mathscr{R}_0 < 1$ , I(t) is monotonically decreasing and approaching to the disease-free equilibrium  $I^* = 0$ . When  $\mathscr{R}_0 > 1$ , I(t) is monotonically approaching to the endemic equilibrium  $I^* = N(1 - 1/\mathscr{R}_0)$ . Figure 3.1 shows three numerical examples of the temporal variation of I(t) when  $\mathscr{R}_0 > 1$ .

#### 3.3 Equilibrium for the model without social insensitivity

In this section, we consider the model (2.3) without the social insensitivity, that is, when  $I_c = 0$ . From (2.3), the model becomes

$$\frac{dI}{dt} = \beta(M)(N-I)I - q(M)I$$

$$\frac{dM}{dt} = \gamma I - \mu M.$$
(3.4)

It is easy to see that the system (3.4) has the disease-free equilibrium  $E_0(0,0)$  and may have the endemic equilibrium  $E_+(I^*, M^*)$  which satisfies that

$$\beta(M^*)(N - I^*) - q(M^*) = 0$$
 and  $\gamma I^* - \mu M^* = 0$ ,

that is,

$$I^* = N - \frac{q(M^*)}{\beta(M^*)}$$
 and  $M^* = \frac{\gamma I^*}{\mu}$ . (3.5)

As for the existence of equilibrium, we can get the following result (Appendix A):

#### Lemma 3.1.

- (i) If  $\mathscr{R}_0 \leq 1$ , there is only the disease-free equilibrium  $E_0(0,0)$ .
- (ii) If  $\mathscr{R}_0 > 1$ , there are two equilibria, the disease-free equilibrium  $E_0(0,0)$  and the endemic equilibrium  $E_+(I^*, M^*)$  with

$$\max\left(0, 1 - \sup_{M} \left[\frac{q(M)}{\beta(M)N}\right]\right) < \frac{I^*}{N} < 1 - \frac{1}{\mathscr{R}_0}.$$

Making use of Lemma 3.1, we can further obtain the following result (see Appendix B):

#### Theorem 3.1.

- (i) If  $\mathscr{R}_0 \leq 1$ , there is the unique equilibrium  $E_0(0,0)$ , which is globally asymptotically stable.
- (ii) If  $\mathscr{R}_0 > 1$ , there are two equilibria  $E_0(0,0)$  and  $E_+(I^*, M^*)$ , of which  $E_0$  is unstable, while  $E_+$  is globally asymptotically stable.

#### 3.4 Equilibrium for the model with social insensitivity

In this section, we are going to consider the model (2.3) with social insensitivity, that is,  $I_c > 0$ . It is easy to see that (2.3) always has the disease-free equilibrium  $E_0(0,0)$  and may have the endemic equilibrium  $(I^*, M^*)$  which satisfies that

$$\beta(M^*)(N-I^*) - q(M^*) = 0$$
 and  $\Gamma(I^*) - \mu M^* = 0.$  (3.6)

As for the existence of equilibrium, we can get the following result (Appendix C):

#### Lemma 3.2.

- (i) If  $\mathscr{R}_0 \leq 1$ , there is the unique equilibrium  $E_0(0,0)$ .
- (ii) If  $1 < \Re_0 \leq (1 \theta_c)^{-1}$  with  $\theta_c := I_c/N$ , there are the disease-free equilibrium  $E_0(0,0)$  and the endemic equilibrium  $E_{+0}(N q_0/\beta_0, 0)$ .
- (iii) If  $\mathscr{R}_0 > (1-\theta_c)^{-1}$ , there are the disease-free equilibrium  $E_0(0,0)$  and the endemic equilibrium  $E_{++}(I^*, M^*)$  with  $I^* > 0$  and  $M^* > 0$ , which is uniquely determined by (3.6) with

$$\max\left(\theta_c, 1 - \sup_M \left[\frac{q(M)}{\beta(M)N}\right]\right) < \frac{I^*}{N} < 1 - \frac{1}{\mathscr{R}_0}$$

The endemic equilibrium  $E_{+0}$  means that the state that people are unconcerned about the disease which persists in the community, while  $E_{++}$  does the state that the persistent disease concern people in the community. With Lemma 3.2, we can obtain the following result (Appendix D):

#### Theorem 3.2.

- (i) If  $\mathscr{R}_0 \leq 1$ , there is the unique equilibrium  $E_0(0,0)$ , which is globally asymptotically stable.
- (ii) If  $1 < \Re_0 \leq (1 \theta_c)^{-1}$ , there are two equilibria  $E_0(0,0)$  and  $E_{+0}(N q_0/\beta_0,0)$ , of which  $E_0$  is unstable, while  $E_{+0}$  is globally asymptotically stable.
- (iii) If  $\mathscr{R}_0 > (1-\theta_c)^{-1}$ , there are two equilibria  $E_0(0,0)$  and  $E_{++}(I^*, M^*)$ , of which  $E_0$  is unstable, while  $E_{++}$  is globally asymptotically stable.

We can also obtain the following about the approaching behavior to  $E_0$  and  $E_{+0}$ :

#### Corollary 3.1.

- (i) If  $\mathscr{R}_0 \leq 1$ , the system approaches  $E_0$  monotonically.
- (ii) If  $1 < \Re_0 \leq (1 \theta_c)^{-1}$ , the system approaches  $E_{+0}$  monotonically.

## Chapter 4

## A specific model

#### 4.1 Modeling for the effect of social response

In this chapter, we consider model (2.1) with the following specific functions for  $\beta(M)$  and q(M):

$$\beta(M) = \frac{\beta_0}{1+aM}; \quad q(M) = q_0 + bM.$$
(4.1)

a and b are the coefficients of the social response effect on the transmission rate and on the recovery rate. The larger a or b means that the social response has stronger effect to work more efficient in avoiding the infection or recovering from the disease.

Then we have

$$\frac{dS}{dt} = -\frac{\beta_0}{1+aM}SI + (q_0 + bM)I$$

$$\frac{dI}{dt} = \frac{\beta_0}{1+aM}SI - (q_0 + bM)I$$

$$\frac{dM}{dt} = \Gamma(I) - \mu M.$$
(4.2)

We consider the community with social insensitivity, that is, when  $\theta_c > 0$ .  $\theta_c$  is corresponding to various factors such as the background of the community, the knowledge about an infectious disease, etc.

With the non-dimensional transformation of variables and parameters given by

$$u = \frac{S}{N}; \quad v = \frac{I}{N}; \quad \tau = q_0 t; \quad \eta = \frac{N\gamma}{q_0}; \quad B = \frac{b}{q_0}; \quad \delta = \frac{\mu}{q_0},$$
 (4.3)

the system (4.2) can be rewritten as follows:

$$\frac{du}{d\tau} = -\frac{\mathscr{R}_0}{1+aM}uv + (1+BM)v$$

$$\frac{dv}{d\tau} = \frac{\mathscr{R}_0}{1+aM}uv - (1+BM)v$$

$$\frac{dM}{d\tau} = G(v) - \delta M,$$
(4.4)



Figure 4.1: Trajectories of model (4.6) with the initial condition (v(0), M(0)) = (0.001, 0.0). Numerical calculation with  $a = 1.0, B = 1.5, \delta = 0.7, \eta = 2.5, \theta_c = 0.1$ . (a)  $\mathscr{R}_0 = 15.0$ . (b)  $\mathscr{R}_0 = 2.0$ .

where

$$G(v) := \begin{cases} 0 & \text{for } v \le \theta_c; \\ \eta(v - \theta_c) & \text{for } v > \theta_c. \end{cases}$$
(4.5)

Parameter  $\eta$  in our model is corresponding to the sensitivity of the community to the prevalence. Note that 0 < u < 1 and 0 < v < 1 with u + v = 1 for any  $t \ge 0$ . When  $\theta_c = 1$ , the above model (4.4) with M(0) = 0 corresponds to the system without social response, which has been considered in Section 3.2.

Since u = 1 - v, the system (4.4) can be reduced to the following 2-dimensional one:

$$\frac{dv}{d\tau} = \frac{\mathscr{R}_0}{1+aM}(1-v)v - (1+BM)v$$

$$\frac{dM}{d\tau} = G(v) - \delta M.$$
(4.6)

#### 4.2 Parameter dependence of endemic equilibrium

From Theorem 3.2, the system (4.6) always has the disease-free equilibrium  $E_0(0,0)$ , while the endemic equilibrium uniquely exists and globally asymptotically stable if and only if  $\mathscr{R}_0 > 1$  (see Figure 4.1). If  $1 < \mathscr{R}_0 \leq (1 - \theta_c)^{-1}$ , the endemic equilibrium is  $E_{+0} = (1 - \mathscr{R}_0^{-1}, 0)$ , while if  $\mathscr{R}_0 > (1 - \theta_c)^{-1}$ , the endemic equilibrium is  $E_{++} = (v^*, M^*)$  with

$$v^* = 1 - \frac{(1 + aM^*)(1 + BM^*)}{\mathscr{R}_0}$$
 and  $v^* = \theta_c + \frac{\delta M^*}{\eta}$ . (4.7)

Thus we can derive

$$M^* = \frac{-\left(a+B+\frac{\delta\mathscr{R}_0}{\eta}\right) + \sqrt{\left(a+B+\frac{\delta\mathscr{R}_0}{\eta}\right)^2 - 4aB(\theta_c\mathscr{R}_0 - \mathscr{R}_0 + 1)}}{2aB}.$$
 (4.8)



Figure 4.2: Bifurcation diagram for system (4.4), using solid curves for stable equilibria and dashed line for unstable equilibrium. Numerical calculation with a = 1.0; B = 1.5;  $\delta = 0.7$ ;  $\eta = 0.9$ ;  $\theta_c = 0.5$ .

We can easily prove that  $0 < v^* < 1$  (Appendix E). Further, we can show the following continuity between  $E_{++}$  and  $E_{+0}$ :

Lemma 4.1.  $E_{++} \rightarrow E_{+0}$  as  $\mathscr{R}_0 \rightarrow 1/(1-\theta_c)$ .

The endemic size  $v^*$  depends on the basic reproduction number  $\mathscr{R}_0$  as shown in Figure 4.2. As expected,  $v^*$  is monotonically increasing in terms of  $\mathscr{R}_0$ . Further, as for its dependence on the other parameters, from the partial derivate of  $v^*$  in terms of a, B,  $\delta$  and  $\eta$ , we can get the following result (Appendix F):

**Theorem 4.1.** When  $1 < \mathscr{R}_0 < (1 - \theta_c)^{-1}$ ,  $v^*$  at  $E_+$  or  $E_{+0}$  is increasing in terms of  $\mathscr{R}_0$ . When  $\mathscr{R}_0 > (1 - \theta_c)^{-1}$ ,  $v^*$  at  $E_{++}$  is increasing in terms of  $\mathscr{R}_0$ ,  $\theta_c$  and  $\delta$ , while  $v^*$  at  $E_{++}$  is decreasing in terms of a, B and  $\eta$ .

We numerically show the parameter dependence of the endemic equilibrium value  $v^*$  in Figure 4.3. The higher efficiency of social response on the transmission rate and the recovery rate, the smaller endemic size at the equilibrium state. In an situation, the community with the higher sensitivity will show the stronger response to the prevalence. For example, increasing the frequency of masking higher than others, the endemic size can be reduced. When  $\mathscr{R}_0 > 1$ , there is a critical value for  $\mathscr{R}_0$ : If  $\mathscr{R}_0 < 1/(1 - \theta_c)$ ,  $v^*$  becomes much larger as  $\mathscr{R}_0$  gets larger. increases drastically with the growth of  $\mathscr{R}_0$ . If  $\mathscr{R}_0 < 1/(1 - \theta_c)$ ,  $v^*$  is independent of  $\theta_c$ .



Figure 4.3: Parameter dependent plots of the endemic equilibrium value  $(v^*)$  of model (4.6). Numerical calculation with a = 1.0; B = 1.5;  $\Re_0 = 5.0$ . (a)  $\eta = 0.9$ ;  $\delta = 0.7$ . (b)  $\theta_c = 0.5$ ;  $\eta = 0.9$ . (c)  $\theta_c = 0.5$ ;  $\delta = 0.7$ .

#### **CASE OF** B = 0

When there is no effect of social response on the recovery rate, for example, with no effective medicine or vaccine for the disease. This is the case of B = 0 in our model:

$$\frac{dv}{d\tau} = \frac{\mathscr{R}_0}{1+aM}(1-v)v - v$$

$$\frac{dM}{d\tau} = G(v) - \delta M.$$
(4.9)

The endemic equilibrium  $E_{++}(v^*, M^*)$  can be easily obtained as follows:

$$M^* = \frac{\{\mathscr{R}_0(1-\theta_c)-1\}\eta}{a\eta + \mathscr{R}_0\delta}; \quad v^* = \theta_c + \frac{\{\mathscr{R}_0(1-\theta_c)-1\}\delta}{a\eta + \mathscr{R}_0\delta}.$$
 (4.10)

It is easy to see that, the endemic size  $v^*$  is decreasing in terms of a, which means if there is no social response effect on the recovery rate, the endemic size  $v^*$  can be reduced by stronger social response effect on the transmission rate. The social response on transmission rate has a negative effect on the endemic size  $v^*$ .

#### CASE OF a = 0

When we consider the case that there is no effect of social response on the disease transmission (a = 0). For example, this is the case with no appropriate protective measure. Our model becomes

$$\frac{dv}{d\tau} = \mathscr{R}_0(1-v)v - (1+BM)v$$

$$\frac{dM}{d\tau} = G(v) - \delta M.$$
(4.11)

The endemic equilibrium  $E_{++}(v^*, M^*)$  can be easily obtained as follows:

$$M^* = \frac{\{\mathscr{R}_0(1-\theta_c)-1\}\eta}{B\eta+\mathscr{R}_0\delta}; \quad v^* = \theta_c + \frac{\{\mathscr{R}_0(1-\theta_c)-1\}\delta}{B\eta+\mathscr{R}_0\delta}.$$



Figure 4.4: Numerical examples of temporal variation by model (4.6) with (v((0), M(0)) = (0.001, 0). Numerical calculation with  $\theta_c = 0.3$ ;  $\mathscr{R}_0 = 4.0$ ; a = 5.0; B = 5.5;  $\delta = 0.5$ . (a)  $\eta = 5.0$ . (b)  $\eta = 0.01$ 

It is easy to see that, the endemic size  $v^*$  is decreasing in terms of B, which means if there is no social response effect on the transmission rate, the endemic size  $v^*$  can be reduced by stronger social response effect on the recovery rate. The social response on recovery rate has a negative effect on the endemic size  $v^*$ .

#### 4.3 Occurrence of oscillatory behavior

We have already gotten the result of Theorem 3.2, which shows that the system (4.2) necessary approaches an equilibrium. Further, from Corollary 3.1, we have found that, expect for the case that  $E_{++}$  exists, the system (4.2) approaches an equilibrium in a monotonic manner. In contrast, when  $E_{++}$  exists, the system (4.2) may show a damped oscillation as shown by Figure 4.4. Since the temporally oscillation of infective population size means a repetition of outbreaks of disease spread, it is worthwhile investigating the dependence of the occurrence of such an oscillatory behavior in the system (4.2). So we consider the eigenvalue for the equilibrium  $E_{++}$ , because such an oscillatory behavior can appear if and only if the eigenvalue is imaginary.

The Jacobian matrix  $J(E_{++})$  for the endemic state  $E_{++}$  of the system (4.6) is given by

$$J(E_{++}) = \begin{pmatrix} -\Phi_1 & -\Phi_2 \\ & & \\ \eta & -\delta \end{pmatrix},$$

where

$$\Phi_1 := \frac{\mathscr{R}_0}{1+aM^*} v^*; \quad \Phi_2 := \left\{ \frac{a\mathscr{R}_0}{(1+aM^*)^2} (1-v^*) + B \right\} v^*.$$
(4.12)

It is easy to see that  $\Phi_1 > 0$  and  $\Phi_2 > 0$  when  $\mathscr{R}_0 > 1$ . The characteristic equation for  $J(E_{++})$  becomes

$$\lambda^2 + (\Phi_1 + \delta)\lambda + \Phi_1\delta + \Phi_2\eta = 0. \tag{4.13}$$

Hence the eigenvalues  $\lambda_1$  and  $\lambda_2$  satisfy

$$\lambda_1 + \lambda_2 = -(\Phi_1 + \delta) < 0; \quad \lambda_1 \lambda_2 = \Phi_1 \delta + \Phi_2 \eta > 0.$$

From (4.7), we obtain the following expression of the discriminant of (4.13):

$$\Delta = \frac{1}{(1+aM^*)^2} (\mathscr{A}M^* + \mathscr{B}), \qquad (4.14)$$

where

$$\begin{split} \mathscr{A} &:= -\frac{1}{aB} \left( a + B + \frac{\mathscr{R}_0 \delta}{\eta} \right) \left\{ \left( \frac{\mathscr{R}_0 \delta}{\eta} - a\delta \right)^2 - 4a\delta \left( B - a - \frac{2\mathscr{R}_0 \delta}{\eta} \right) \right\} \\ &+ 2 \left( \frac{\mathscr{R}_0 \delta}{\eta} - a\delta \right) \left( \mathscr{R}_0 \theta_c - \delta \right) - 4\delta (B - a + 2a\mathscr{R}_0 - 4a\mathscr{R}_0 \theta_c) - 4a\theta_c \eta (B - a), \\ \mathscr{B} &:= -\frac{1}{aB} \left( -\mathscr{R}_0 + \mathscr{R}_0 \theta_c + 1 \right) \left\{ \left( \frac{\mathscr{R}_0 \delta}{\eta} - a\delta \right)^2 - 4a\delta \left( B - a - \frac{2\mathscr{R}_0 \delta}{\eta} \right) \right\} \\ &+ (\mathscr{R}_0 \theta_c - \delta)^2 - 4\theta_c \eta (B - a + 2a\mathscr{R}_0 - 2a\mathscr{R}_0 \theta_c). \end{split}$$

For any initial state (v(0), M(0)), where v(0) > 0 and M(0) = 0, if  $\Delta < 0$ , then (v(t), M(t)) approaches the equilibrium  $E_{++}$  with a damped oscillation. If  $\Delta \ge 0$ , (v(t), M(t)) simply approaches the equilibrium  $E_{++}$ .

From Theorem 4.1, we know that the social insensitivity has a positive effect on the endemic size. And from Theorem 3.2, we know that if the endemic equilibrium  $E_{++}$  exists,  $E_{++}$  is globally asymptotically stable. However, the system near  $E_{++}$  may show different behaviors, which is dependent on the solutions of the characteristic equation for  $J(E_{++})$ .

When  $\theta_c = 0$ , from (4.8), we have

$$M_{\theta_c=0}^* = \frac{-(a+B+\frac{\delta\mathscr{R}_0}{\eta}) + \sqrt{(a+B+\frac{\delta\mathscr{R}_0}{\eta})^2 + 4aB(\mathscr{R}_0-1))}}{2aB}.$$

When  $\theta_c \ll 1$ , we define

$$H := \frac{\delta}{\eta} \frac{\partial M^*}{\partial \theta_c} \Big|_{\theta_c = 0}$$
$$= -\frac{1}{\frac{\eta}{\delta \mathscr{R}_0} \left\{ a(1 + BM^*_{\theta_c = 0}) + B(1 + aM^*_{\theta_c = 0}) \right\} + 1}.$$
(4.15)

It is easy to see that -1 < H < 0. From (4.12), we have

$$\Phi_1 = \phi_{11}\theta_c + \phi_{12} + o(\theta_c); \quad \Phi_1 = \phi_{21}\theta_c + \phi_{22} + o(\theta_c),$$

where

$$\begin{split} \phi_{11} &= \frac{\delta}{\eta} \frac{\mathscr{R}_0 M_{\theta_c=0}^*}{1 + a M_{\theta_c=0}^*}; \\ \phi_{12} &= \frac{\mathscr{R}_0}{1 + a M_{\theta_c=0}^*} \left( 1 + \frac{H}{1 + a M_{\theta_c=0}^*} \right); \\ \phi_{21} &= \left\{ \frac{a (1 + B M_{\theta_c=0}^*)}{1 + a M_{\theta_c=0}^*} + B \right\} \frac{\delta}{\eta} M_{\theta_c=0}^*; \\ \phi_{22} &= \frac{a (B - a) M_{\theta_c=0}^*}{(1 + a M_{\theta_c=0}^*)^2} H + \left\{ \frac{a (1 + B M_{\theta_c=0}^*)}{1 + a M_{\theta_c=0}^*} + B \right\} (1 + H). \end{split}$$

We can easily see that  $\phi_{11} > 0$ ,  $\phi_{12} > 0$  and  $\phi_{21} > 0$ . From (4.14), we have

$$\Delta = \left\{ 2\phi_{12}(\phi_{11} - \delta) - 4\eta\phi_{22} \right\} \theta_c + (\phi_{11} - \delta)^2 - 4\eta\phi_{21} + o(\theta_c).$$

If  $\Delta(\theta_c = 0) > 0$ ,  $\Delta > 0$  for  $\theta_c \ll 1$ . If  $\Delta(\theta_c = 0) < 0$ ,  $\Delta < 0$  for  $\theta_c \ll 1$ . When  $\Delta(\theta_c = 0) > 0$ , if  $2\phi_{12}(\phi_{11} - \delta) - 4\eta\phi_{22} < 0$ ,  $\Delta$  is decreasing in terms of  $\theta_c$  (for  $\theta_c \ll 1$ ), thus the existence of social insensitivity works to generate the oscillatory. When  $\Delta(\theta_c =) < 0$ , if  $2\phi_{12}(\phi_{11} - \delta) - 4\eta\phi_{22} > 0$ ,  $\Delta$  is decreasing in terms of  $\theta_c$  (for  $\theta_c \ll 1$ ), thus the existence of social insensitivity works to promote the oscillatory.

However, the dependence on  $\theta_c$  is not simple. For lager  $\theta_c$ , it can promote the occurrence of damped oscillation (see Figure 4.5 (c-1) and Figure 4.6 (a-2)) but can also suppress the occurrence of damped oscillation (see Figure 4.5 (a), (b), (c-2) and (c-3)). As for the larger  $\delta$ , it can promote the occurrence of damped oscillation, but it can also suppress the occurrence of damped oscillation (see Figure 4.6). As shown in Figure 4.5, the larger  $\eta$  promotes the occurrence of damped oscillation. From Figure 4.7, for positive a and B, we can see that if social response on the infection rate and recovery rate becomes stronger, damped oscillation is more likely to occur. If the infection rate and the recovery rate are more sensitive to the social response, the oscillatory approach is more likely to occur. When there is no social response effect on the infection rate, that is, B = 0, if the social insensitivity exists ( $\theta_c > 0$ ), larger  $\eta$  is more likely to cause damped oscillation. If there is no social insensitivity, if and only if  $\delta > \Re_0$ , larger  $\eta$  is more likely to cause damped oscillation. Otherwise, larger  $\eta$  is more likely to suppress damped oscillation (Appendix G).

When there is no social response effect on the recovery rate, that is, a = 0, if the social insensitivity exists  $(\theta_c > 0)$ , larger  $\eta$  is more likely to cause damped oscillation. If there is no social insensitivity, if and only if  $\delta < 4(\mathscr{R}_0 - 1)$ , larger  $\eta$  is more likely to cause damped oscillation. Otherwise, larger  $\eta$  is more likely to suppress damped oscillation (Appendix H).



Figure 4.5: Region plots of the occurrence of damped oscillation of model (4.6). Numerical calculation with (a, b-1)  $\delta = 1.0$ ; (a, b-2)  $\delta = 2.0$ ; (a, b-3)  $\delta = 3.0$ . (a)  $\mathscr{R}_0 = 3.0$ ; (b)  $\mathscr{R}_0 = 3.5$ ; (c)  $\mathscr{R}_0 = 5.5$ . Commonly, a = 1.0 and B = 1.5.



Figure 4.6: Region plots of the occurrence of damped oscillation of model (4.6). Numerical calculation with (a, b-1)  $\eta = 0.1$ ; (a, b-2)  $\eta = 0.2$ ; (a, b-3)  $\eta = 0.3$ . (a)  $\mathscr{R}_0 = 3.0$ ; (b)  $\mathscr{R}_0 = 3.5$ ; (c)  $\mathscr{R}_0 = 5.5$ . Commonly, a = 1.0 and B = 1.5.



Figure 4.7: Region plots of the occurrence of damped oscillation of model (4.6). Numerical calculation with (a, b-1)  $\theta_c = 0$ ; (a, b-2)  $\theta_c = 0.2$ . (a) B = 1.5; (b) a = 1.0. Commonly,  $\delta = 3.5$  and  $\Re_0 = 3.0$ .



Figure 4.8: Region plots of the occurrence of damped oscillation of model (4.6). Numerical calculation with B = 1.5; a = 1.0;  $\Re_0 = 3.0$ . (a)  $\theta_c = 0.0$ . (b)  $\theta_c = 0.3$ .



Figure 4.9: Region plots of the occurrence of damped oscillation of model (4.6). Numerical calculation with a = 1.0; B = 1.5;  $\mathcal{R}_0 = 3.0$ .  $\eta = 0.1$  for (a);  $\eta = 0.2$  for (b);  $\mathcal{R}_0 = 3.0$  for (1);  $\mathcal{R}_0 = 3.5$  for (2);  $\mathcal{R}_0 = 5.5$  for (3).

# Chapter 5 Conclusion

To analysis the effect of social response on epidemic dynamics, we propose a generic SIS+M model to characterize the social behavior to a transmissible disease. In our model, we consider the social response effect on the infection rate and the recovery rate. It is clear that if there is no social response, the system shows a monotonic approach and it will go to a stable state as time goes on. When social response exists in the epidemic dynamics, the system will also approach to the endemic equilibrium but in different ways. Another important index that we considered is social insensitivity. In the epidemic dynamics, for sufficient strong social insensitivity, the final size of the endemic equilibrium will go to a constant and become maximal.

In our model with specific functions for the infection rate and the recovery rate with the effect of social response, it is very reasonable to see that the larger basic reproduction number, the larger size at the endemic equilibrium state. We also show that if the community is more insensitive to the disease, the endemic size becomes larger. As for the sensitivities to social response of the infection rate and the recovery rate, the stronger sensitivity reduces the endemic size. Our result also shows that the stronger decay rate of social response reduces the endemic size, and the stronger social sensitivity causes the larger size at the endemic equilibrium state.

Though the endemic equilibrium is always stable as long as it exits, the behavior of approaching near it shows two different ways: approaching monotonically or damped oscillatory approaching. In our numerical calculation, we found that the stronger decay rate of social response is more likely to cause damped oscillation. However, this will not always occur if there is no social response effect on the coefficient of the infection rate (recovery rate) and the social insensitivity. In the case of model with social insensitivity, it is clear that the larger social insensitivity can cause the larger endemic size. But, the effect of social insensitivity to the approaching behavior is not so simple. For example, social response can suppress damped oscillation in some cases while promote damped oscillation in other cases.

Although our SIS+M model is based on the simplest model: SIS model, the approaching behavior near to the endemic equilibrium is not simple, especially on the effect of social insensitivity. Based on our result from this paper, we expect that our

further work would be more useful to explain the nature of a transmissible disease spread and the effect of social behavior change.

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## Appendix A

## Proof of Lemma 3.1

The endemic equilibrium exists if  $I^*$  and  $M^*$  are positive from biological meaning of I and M. From (3.5),

$$N - \frac{q(M^*)}{\beta(M^*)} > 0.$$

Since

$$\frac{d\left(N - \frac{q(M)}{\beta(M)}\right)}{dM} = -\frac{q'(M)\beta(M) - q(M)\beta'(M)}{\beta(M)^2} < 0,$$

Thus  $N - \frac{q(M)}{\beta(M)}$  is a decreasing function, and  $I^* > 0$  when  $N - q_0/\beta_0 > 0$ , that is,  $\Re_0 > 1$ .  $I^*$  satisfies:

$$\max\left(0, N - \sup\left[\frac{q(M)}{\beta(M)}\right]\right) < I^* < N - \frac{q_0}{\beta_0}.$$

## Appendix B

## Proof of Theorem 3.1

To analyze the stability of the equilibrium for the system (3.4), we can use the phase plane of (I, M), in which we can identify the vector flow for the system, as shown in Figure B.1. If  $\mathscr{R}_0 \leq 1$ , (I(t), M(t)) approaches the unique equilibrium  $E_0$ , thus  $E_0$  is globally asymptotically stable. If  $\mathscr{R}_0 > 1$ , when  $I(t) > N - q_0/\beta_0$ , we have dI/dt < 0, so that the trajectory must go down. when  $M(t) > (N - q_0/\beta_0)\mu/\gamma$ , then dM/dt < 0, and the trajectory will go to the left side. Since there is no other stable equilibrium than  $E_+$ , thus all initial condition  $(I(0), M(0)) \in U := (0, N - q_0/\beta_0) \times (0, (N - q_0/\beta_0)\mu/\gamma)$  will remain in U and the trajectory can not go to infinitive part. (I(t), M(t)) is approaching to  $E_+$  and will go far away from  $E_0$ . If  $\mathscr{R}_0 > 1$ , the trajectory near  $E_+$  will asymptotically approach to  $E_+$ .

Now we discuss about the local stability of  $E_+$ . The Jacobian matrix  $J(E_+)$  for the equilibrium  $E_+$  is given by



Figure B.1: Rough sketch of the vector flow in the (M, I)-phase plane for the system (3.4) with the nullclines. (a)  $\mathscr{R}_0 \leq 1$ . (b)  $\mathscr{R}_0 > 1$ .

From (3.5),

$$J(E_+) = \begin{pmatrix} -\Phi_1 & -\Phi_2 \\ & & \\ \gamma & -\mu \end{pmatrix},$$

where

$$\Phi_1 := \beta(M^*)I^*; \quad \Phi_2 := -\left\{\frac{q(M^*)}{\beta(M^*)}\beta'(M^*) - q'(M^*)\right\}I^*.$$

Since  $\beta(M) > 0$ , q(M) > 0 for any  $M \ge 0$ , and  $\beta(M)$  is a decreasing function while q(M) is an increasing function of M, thus  $\beta'(M) < 0$  and q'(M) > 0. Therefore  $\Phi_1 > 0$ ,  $\Phi_2 > 0$ . The characteristic equation for  $J(E_+)$  becomes:

$$\lambda^2 + (\Phi_1 + \mu)\lambda + \Phi_1\mu + \Phi_2\gamma = 0. \tag{B.1}$$

Hence the eigenvalues  $\lambda_1$  and  $\lambda_2$  satisfy:

$$\lambda_1 + \lambda_2 = -(\Phi_1 + \mu) < 0; \quad \lambda_1 \lambda_2 = \Phi_1 \mu + \Phi_2 \gamma > 0.$$

The discriminant of (B.1) can be derived as

$$\Delta = (\Phi_1 + \mu)^2 - 4(\Phi_1 \mu + B\gamma) = (\Phi_1 - \mu)^2 - 4\Phi_2\gamma.$$
(B.2)

If  $\Delta > 0$ ,  $\lambda_1$  and  $\lambda_2$  are both real and negative. If  $\Delta < 0$ ,  $\lambda_1$  and  $\lambda_2$  are imaginary, we have  $Re\lambda_1 = Re\lambda_2 < 0$ . Therefore, all the eigenvalues of  $J(E_+)$  have negative real parts. Thus, if the equilibrium  $E_+$  exists,  $E_+$  is locally asymptotically stable.

## Appendix C

## Proof of Lemma 3.2

The endemic equilibrium exists if  $I^*$  and  $M^*$  are positive from biological meaning of I and M. From (3.6),

$$N - \frac{q(M^*)}{\beta(M^*)} > I_c.$$

Since

$$\frac{d\left(N - \frac{q(M)}{\beta(M)}\right)}{dM} = -\frac{q'(M)\beta(M) - q(M)\beta'(M)}{\beta(M)^2} < 0,$$

Thus  $N - \frac{q(M)}{\beta(M)}$  is a decreasing function, and  $I^* > I_c$  when  $N - q_0/\beta_0 > I_c$ , that is,  $\mathscr{R}_0 > (1 - \theta_c)^{-1}$ .  $I^*$  satisfies:

$$\max\left(I_c, N - \sup\left[\frac{q(M)}{\beta(M)}\right]\right) < I^* < N - \frac{q_0}{\beta_0}.$$

## Appendix D

## Proof of Theorem 3.2

gTo analyze the stability of the equilibrium for the system (2.3), we can use the phase plane of (I, M) as shown in Figure D.1. Let set up the initial condition I(0) > 0 and M(0) = 0. If  $\mathscr{R}_0 \leq 1$ , (I(t),M(t)) approaches the unique equilibrium  $E_0$ . If  $1 < \mathscr{R}_0 \leq (1 - \theta_c)^{-1}$ , (I(t),M(t)) is approaching to  $E_{+0}(N - q_0/\beta_0, 0)$  and will go far away from  $E_0(0,0)$ . If  $\mathscr{R}_0 > (1-\theta_c)^{-1}$ , there are two equilibria  $E_0(0,0)$  and  $E_{++}(I^*,M^*)$ . It is easy to find an invariant neighbourhood  $U_+$  of  $E_{++}$  Since there is no other stable equilibrium that  $E_{++}$ , thus for any initial condition, the trajectory will not go to infinitive part, and (I(t),M(t)) is approaching to  $E_{++}(I^*,M^*)$  and going far way from  $E_0(0,0)$ . If  $\mathscr{R}_0 > (1 - \theta_c)^{-1}$ , the trajectory near  $E_{++}$  will asymptotically approach to  $E_{++}$  (see Appendix D). Now we discuss the local stability of  $E_{++}$ . the Jacobian matrix of  $E_{++}$  is given by:



Figure D.1: Rough sketch of the vector flow in the (M, I)-phase plane for the system (2.3) with the nullclines. (a)  $\mathscr{R}_0 \leq 1$ . (b)  $1 < \mathscr{R}_0 \leq (1 - \theta_c)^{-1}$ . (c)  $\mathscr{R}_0 > (1 - \theta_c)^{-1}$ .

From the equation (3.6),

$$J(E_{++}) = \begin{pmatrix} -\Phi_{1+} & -\Phi_{2+} \\ & & \\ \gamma & -\mu \end{pmatrix},$$

where

$$\Phi_{1+} := \beta(M^*)I^*; \quad \Phi_{2+} := -\left\{\frac{q(M^*)}{\beta(M^*)}\beta'(M^*) - q'(M^*)\right\}I^*.$$

Since  $\beta'(M) < 0$  and q'(M) > 0,  $\Phi_{1+} > 0$  and  $\Phi_{2+} > 0$ . The characteristic equation for  $J(E_{++})$  becomes:

$$\lambda^{2} + (\Phi_{1+} + \mu)\lambda + \Phi_{1+}\mu + \Phi_{2+}\gamma = 0.$$
 (D.1)

Therefore, the eigenvalues  $\lambda_{+1}$  and  $\lambda_{+2}$  satisfy:

$$\lambda_{+1} + \lambda_{+2} = -(\Phi_{1+} + \mu) < 0; \quad \lambda_{+1}\lambda_{+2} = \Phi_{1+}\mu + \Phi_{2+}\gamma > 0.$$

The discriminant of (D.1) is given by

$$\Delta = (\Phi_{1+} + \mu)^2 - 4(\Phi_{1+}\mu + B\gamma) = (\Phi_{1+} - \mu)^2 - 4\Phi_{2+}\gamma.$$
(D.2)

If  $\Delta > 0$ ,  $\lambda_{+1}$  and  $\lambda_{+2}$  are both real and negative. If  $\Delta < 0$ ,  $\lambda_{+1}$  and  $\lambda_{+2}$  are imaginary, we have  $Re\lambda_{+1} = Re\lambda_{+2} < 0$ . Thus, all the eigenvalues of  $J(E_{++})$  have negative real parts. Thus, if the equilibrium  $E_{++}$  exists, then  $E_{++}$  is locally asymptotically stable.

## Appendix E

## **Proof of** $0 < v^* < 1$

Proof 1 From the equations (4.7), we have

$$\theta_c + \frac{\delta M^*}{\eta} = 1 - \frac{(1 + aM^*)(1 + BM^*)}{\mathscr{R}_0}.$$
 (E.1)

Define

$$f(x) = aBx^{2} + \left(a + B + \frac{\mathscr{R}_{0}\delta}{\eta}\right)x + (\mathscr{R}_{0}\theta_{c} - \mathscr{R}_{0} + 1),$$

where  $a, B, \delta, \eta, \theta_c$  and  $\mathscr{R}_0$  are defined as (4.3) and (3.1).  $f(M^*) = 0$  is mathematically equivalent to (E.1). It is easily shown that  $f(M^*) = 0$  has a unique positive root. Then from (4.7) we have

$$v^* = \theta_c + \frac{\delta M^*}{\eta} > 0.$$

Thus from (E.1),

$$v^* = 1 - \frac{(1+aM^*)(1+BM^*)}{\mathscr{R}_0} > 0.$$
 (E.2)

Next, we will show that  $v^* < 1$ . f(x) is negative for  $0 < x < M^*$  and positive for  $x > M^*$ . To prove  $v^* = \theta_c + (\delta M^*)/\eta < 1$ , we need to prove

$$\frac{\eta}{\delta}(1-\theta_c) > M^*. \tag{E.3}$$

Since

$$f\left(\frac{\eta}{\delta}(1-\theta_c)\right) = aB\left(\frac{\eta}{\delta}\right)^2 (1-\theta_c)^2 + \left(a+B+\frac{\mathscr{R}_0\delta}{\eta}\right)\frac{\eta}{\delta}(1-\theta_c) + (\mathscr{R}_0\theta_c - \mathscr{R}_0 + 1)$$
  
> 0,

thus the inequality (E.3) holds. As a result, we have shown that  $v^* < 1$ .

# Appendix F

## Proof of Theorem 4.1

We derive the partial derivates of (4.7) in terms of a,

$$\begin{split} \frac{\partial v^*}{\partial a} &= -\frac{1}{\mathscr{R}_0} \bigg\{ BM^{*2} + 2aBM^* \frac{\partial M^*}{\partial a} + M^* + (a+B) \frac{\partial M^*}{\partial a} \bigg\},\\ \frac{\partial v^*}{\partial a} &= \frac{\delta}{\eta} \frac{\partial M^*}{\partial a}, \end{split}$$

Thus we can derive

$$\left(\frac{2aB}{\mathscr{R}_0}\frac{\eta}{\delta}M^* + \frac{a+B}{\mathscr{R}_0}\frac{\eta}{\delta} + 1\right)\frac{\partial v^*}{\partial a} = -\frac{1}{\mathscr{R}_0}BM^{*2} - \frac{1}{\mathscr{R}_0}M^*.$$

We can also derive

$$\left(\frac{2aB}{\mathscr{R}_0}\frac{\eta}{\delta}M^* + \frac{a+B}{\mathscr{R}_0}\frac{\eta}{\delta} + 1\right)\frac{\partial v^*}{\partial B} = -\frac{1}{\mathscr{R}_0}aM^{*2} - \frac{1}{\mathscr{R}_0}M^*.$$

Thus  $\partial v^*/\partial a < 0$  and  $\partial v^*/\partial B < 0$ . We derive the partial derivates of (4.7) in terms of  $\delta$ ,

$$\begin{split} &\frac{\partial v^*}{\partial a} = -\frac{1}{\mathscr{R}_0} \Big\{ a(1+BM^*) + B(1+aM^*) \Big\} \frac{\partial M^*}{\partial \delta}, \\ &\frac{\partial v^*}{\partial \delta} = \frac{M^*}{\eta} + \frac{\delta}{\eta} \frac{\partial M^*}{\partial \delta}, \end{split}$$

Thus we can derive

$$\left[\frac{\eta}{\delta}\frac{1}{\mathscr{R}_0}\left\{a(1+BM^*)+B(1+aM^*)\right\}+1\right]\frac{\partial v^*}{\partial \delta}=\frac{M^*}{\mathscr{R}_0\delta}\left\{a(1+BM^*)+B(1+aM^*)\right\},$$

then  $\partial v^*/\partial \delta > 0$ . We can derive the partial derivates of  $v^*$  in terms of  $\eta$ ,  $\mathscr{R}_0$  and  $\theta_c$ :

$$\begin{split} & \left[\frac{\eta}{\delta}\frac{1}{\mathscr{R}_{0}}\left\{a(1+BM^{*})+B(1+aM^{*})\right\}+1\right]\frac{\partial v^{*}}{\partial \eta}=-\frac{M^{*}}{\mathscr{R}_{0}\eta}\left\{a(1+BM^{*})+B(1+aM^{*})\right\},\\ & \left[\frac{\eta}{\delta}\frac{1}{\mathscr{R}_{0}}\left\{a(1+BM^{*})+B(1+aM^{*})\right\}+1\right]\frac{\partial v^{*}}{\partial \mathscr{R}_{0}}=\frac{1}{\mathscr{R}_{0}^{-2}}\left\{a(1+BM^{*})+B(1+aM^{*})\right\},\\ & \left[\frac{\eta}{\delta}\frac{1}{\mathscr{R}_{0}}\left\{a(1+BM^{*})+B(1+aM^{*})\right\}+1\right]\frac{\partial v^{*}}{\partial \theta_{c}}=\frac{\eta}{\delta}\frac{1}{\mathscr{R}_{0}}\left\{a(1+BM^{*})+B(1+aM^{*})\right\},\end{split}$$

thus  $\partial v^* / \partial \eta > 0$ ,  $\partial v^* / \partial \mathscr{R}_0 > 0$  and  $\partial v^* / \partial \theta_c > 0$ .

## Appendix G

# Oscillatory behavior in the case of B = 0

When B = 0, we have

$$v^* = 1 - \frac{1 + aM^*}{\mathscr{R}_0}$$
 and  $v^* = \theta_c + \frac{\delta M^*}{\eta}$ .

From (4.12), we have

$$\Phi_1 = \frac{v^*}{1 - v^*}; \quad \Phi_2 = \frac{a}{\mathscr{R}_0} \Phi_1.$$

Therefore

$$\Delta = \Phi_1^2 - 2\left(\delta + \frac{2a\eta}{\mathscr{R}_0}\right)\Phi_1 + \delta^2. \tag{G.1}$$

The discriminant of (G.1) is given by

$$D = 4\left(\frac{2a\eta}{\Re_0}\right)^2 \left\{1 + \Re_0\left(\frac{\delta}{a\eta}\right)\right\} > 0.$$

Thus,  $\Delta < 0$  for  $\exists \Phi_1 > 0$  if and only if

$$0 < \Phi_{1-}^* < \Phi_1 < \Phi_{1+}^*,$$

where

$$\Phi_{1\mp}^* := \frac{2a\eta}{\mathscr{R}_0} \left[ 1 + \frac{\mathscr{R}_0}{2} \left( \frac{\delta}{a\eta} \right) \mp \sqrt{1 + \mathscr{R}_0 \left( \frac{\delta}{a\eta} \right)} \right].$$

From (4.10)

$$\Phi_1 = \frac{\theta_c + (\mathscr{R}_0 - 1)(\delta/a\eta)}{1 - \theta_c + (\delta/a\eta)}.$$
(G.2)

From (G.2), we can obtain

$$\theta_{-}^{*} = \frac{1}{1 + \Phi_{1-}^{*}} \left[ \Phi_{1-}^{*} + \frac{\delta}{a\eta} \left\{ \Phi_{1-}^{*} - (\mathscr{R}_{0} - 1) \right\} \right]; \quad \theta_{+}^{*} = \frac{1}{1 + \Phi_{1+}^{*}} \left[ \Phi_{1+}^{*} + \frac{\delta}{a\eta} \left\{ \Phi_{1+}^{*} - (\mathscr{R}_{0} - 1) \right\} \right]$$

**Lemma G.1.** Oscillatory approach exists if and only if  $\theta_{-}^* < \theta_c < \theta_{+}^*$ .

Let  $\Delta = 0$ . When  $\theta_c = 0$ ,

$$\{\mathscr{R}_0\delta + 4(1-\mathscr{R}_0)\}(a\eta)^2 - 2\delta\{\mathscr{R}_0(\mathscr{R}_0 - 1 - \delta) + 2(\mathscr{R}_0 - 1)\}(a\eta) + \mathscr{R}_0\delta(\mathscr{R}_0 - 1 - \delta)^2 = 0$$
(G.3)

When  $\mathscr{R}_0\delta - 4(\mathscr{R}_0 - 1) < 0$ , that is  $\delta < 4(1 - 1/\mathscr{R}_0)$ , then there is a unique positive root for equation (G.3). When  $\delta > 4(1 - 1/\mathscr{R}_0)$ , if

$$\mathscr{R}_0(\mathscr{R}_0 - 1 - \delta) + 2(\mathscr{R}_0 - 1) < 0,$$

that is if

$$\delta > \mathscr{R}_0 + 1 - 2/\mathscr{R}_0,$$

there is no positive root for equation (G.3). If

$$\delta < \mathscr{R}_0 + 1 - 2/\mathscr{R}_0,$$

and

$$4(1 - 1/\Re_0) < \Re_0 + 1 - 2/\Re_0,$$

Thus, if  $\mathscr{R}_0 > 2$  and

$$4(1-1/\mathscr{R}_0) < \delta < \mathscr{R}_0 + 1 - 2/\mathscr{R}_0,$$

we have: if  $\Delta < 0$ , there is no positive root for equation (G.3); if  $\Delta > 0$  there are two positive roots for equation (G.3). Since  $\mathscr{R}_0 > 4(1 - 1/\mathscr{R}_0)$  when  $\mathscr{R}_0 > 2$ , thus, when  $\mathscr{R}_0 > 2$ , if  $4(1 - 1/\mathscr{R}_0) < \delta < \mathscr{R}_0$  there are two positive real roots for equation (G.3). If  $\delta > \mathscr{R}_0$ , there is no root.

When  $\theta_c > 0$ , then

$$\frac{\mathscr{R}_0}{1+aM^*} - 1 = \frac{a\eta\theta_c + \mathscr{R}_0\delta - \delta}{\delta + a\eta - a\eta\theta_c}.$$

By substituting  $M^*$  into  $\Delta$  and letting  $\Delta = 0$  we obtain

$$\{a\theta_c\eta + \mathscr{R}_0\delta - \delta - \delta(\delta + a\eta - a\eta\theta_c)\}^2 = \frac{4a\eta}{\mathscr{R}_0}(a\eta\theta_c + \mathscr{R}_0\delta - \delta)(\delta + a\eta - a\eta\theta_c)$$

Then

$$k_1(a\eta)^3 + k_2(a\eta)^2 + k_3(a\eta) = k_4,$$
 (G.4)



Figure G.1: Bifurcation diagram for model (4.9) with a = 1.0 and  $\theta_c = 0$ .



Figure G.2: Numerical examples for model (4.9) with a = 1.0;  $\theta_c = 0$ ;  $\mathscr{R}_0 = 4.0$ . (a)  $\delta = 2.0$ . (b)  $\delta = 3.5$ . (c)  $\delta = 4.5$ .

where

$$\begin{split} k_1 &= 4\theta_c(\theta_c - 1); \\ k_2 &= (1 - \theta_c)^2 \mathscr{R}_0 \delta^2 + 2\{ [\theta_c \mathscr{R}_0 (1 + \theta_c) + 2(1 - \mathscr{R}_0 - 2\theta_c)] \delta + \theta_c^2 \mathscr{R}_0 \\ &= (1 - \theta_c)^2 \mathscr{R}_0 \delta^2 - 2(1 - \theta_c)(\theta_c + 2) \bigg\{ \mathscr{R}_0 - \frac{2(1 - 2\theta_c)}{(1 - \theta_c)(\theta_c + 2)} \bigg\} \delta + \theta_c^2 \mathscr{R}_0; \\ k_3 &= 2\delta[(1 - \theta_c) \mathscr{R}_0 \delta^2 - \{ 2(\mathscr{R}_0 - 1) + \mathscr{R}_0 (1 - \theta_c)(\mathscr{R}_0 - 1) + \theta_c \mathscr{R}_0 \} \delta + \theta_c \mathscr{R}_0 (\mathscr{R}_0 - 1)]; \\ k_4 &= -\delta^2 \mathscr{R}_0 (\mathscr{R}_0 - 1 - \delta)^2. \end{split}$$

Define

$$g(x) = x(k_1x^2 + k_2x + k_3).$$

 $g(x) = k_4$  has positive solutions is corresponding to that equation (G.4) has positive roots. Since  $k_1 < 0$  and  $k_4 < 0$ , it is easily to see that there is at least one positive root.

$$g'(x) = 3k_1x^2 + 2k_2x + k_3. (G.5)$$

If and only if g'(x) = 0 has two different real roots  $a\eta_1$ ,  $a\eta_2$  (let us assume  $a\eta_2 > a\eta_1$ ),  $a\eta_1$  is postive and  $a\eta_1$ ,  $a\eta_2$  satisfy

$$g(a\eta_1) < k_4 < g(a\eta_2),$$

then equation (G.4) has three positive roots. From (G.5), g'(x) = 0 has two real roots when

$$k_2^2 - 3k_1k_3 > 0. (G.6)$$

$$g'(a\eta_1) = 3k_1(a\eta_1)^2 + 2k_2(a\eta_1) + k_3 = 0,$$

If positive  $a\eta_1$  exists and satisfies

$$g(a\eta_1) = \frac{1}{9k_1} \bigg\{ (6k_1k_3 - 2k_2^2)a\eta_1 - k_2k_3 \bigg\},\$$

then equation (G.4) has three positive roots. From (G.6), we have

$$a\eta_1 < \frac{k_2k_3 + 9k_1k_4}{2(3k_1k_3 - k_2^2)} < a\eta_2.$$

See Figure G.3.



Figure G.3: (a) Bifurcation diagram for model (4.9) with  $\theta_c = 0.5$  and  $\delta = 5.0$ . Numerical examples with  $\mathscr{R}_0 = 30$ ; (b)  $\delta = 25.0$ ; (c)  $\delta = 35.0$ ;.

## Appendix H

# Oscillatory behavior in the case of a = 0

When a = 0, then

$$\Phi_1 = \mathscr{R}_0 v^*, \quad \Phi_2 = B v^*.$$

And we have

$$v^* = \frac{\theta_c B\eta + \delta(\mathscr{R}_0 - 1)}{\mathscr{R}_0 \delta + B\eta}.$$

Therefore

$$\Delta = \mathscr{R}_0^2 v^{*2} - 2(\mathscr{R}_0 \delta + 2B\eta) v^* + \delta^2.$$
(H.1)

The discriminant of (H.1) is given by

$$D = 4\{(2B\eta)^2 + 4B\eta \mathscr{R}_0 \delta\} > 0.$$

Thus,  $\Delta < 0$  for  $\exists \ v^* > 0$  if and only if

$$0 < v_{-}^{*} < v_{+}^{*} < 1, \tag{H.2}$$

where

$$v_{\mp}^* := \frac{(\mathscr{R}_0 \delta + 2B\eta) \mp \sqrt{(2B\eta)^2 + 4B\eta \mathscr{R}_0 \delta}}{\mathscr{R}_0^2}.$$

From (H.2), we can obtain

$$\theta_{-}^{*} = \frac{(\mathscr{R}_{0}\delta + B\eta)v_{-}^{*} + \delta(1 - \mathscr{R}_{0})}{B\eta}; \quad \theta_{+}^{*} = \frac{(\mathscr{R}_{0}\delta + B\eta)v_{+}^{*} + \delta(1 - \mathscr{R}_{0})}{B\eta}$$

**Lemma H.1.** Oscillatory approach exists if and only if  $\theta_{-}^* < \theta_c < \theta_{+}^*$ .



Figure H.1: (a) Bifurcation diagram for model (4.11) with  $\theta_c = 0$  and B = 1.5. Numerical examples for model (4.11) with (a)  $\delta = 2.0$ ; (b)  $\delta = 10.0$ .

Let  $\Delta = 0$ , then

$$\{\kappa_1(B\eta)^2 + \kappa_2(B\eta) + \kappa_3\}(B\eta) = \kappa_4,\tag{H.3}$$

where

$$\begin{split} \kappa_1 &= 4\theta_c; \\ \kappa_2 &= -\delta^2 + \{4(\mathscr{R}_0 - 1) + 6\theta_c \mathscr{R}_0\}\delta - (\theta_c \mathscr{R}_0)^2; \\ \kappa_3 &= 2\mathscr{R}_0 \delta\{-\delta^2 + (3(\mathscr{R}_0 - 1) + \theta_c \mathscr{R}_0)\delta - \theta_c \mathscr{R}_0(\mathscr{R}_0 - 1)\}; \\ \kappa_4 &= \mathscr{R}_0^2 \delta^2 (\mathscr{R}_0 - 1 - \delta)^2. \end{split}$$

When  $\theta_c = 0$ , then

$$\{4(\mathscr{R}_0-1)-\delta\}(B\eta)^2 + 2\mathscr{R}_0\delta\{3(\mathscr{R}_0-1)-\delta\}(B\eta) - \mathscr{R}_0^2\delta(\mathscr{R}_0-1-\delta)^2 = 0.$$

Since  $\mathscr{R}_0^2 \delta(\mathscr{R}_0 - 1 - \delta)^2 > 0$ , Thus, when  $\delta < 4(\mathscr{R}_0 - 1)$ , there is a unique positive root for  $\Delta = 0$ . When  $\delta > 4(\mathscr{R}_0 - 1)$ , then  $\delta > 3(\mathscr{R}_0 - 1)$  is always hold, thus, there is no positive root.

For  $\theta_c > 0$ , since  $\kappa_1 > 0$  and  $\kappa_4 > 0$ , there is at least one positive root. We define

$$h(x) = x(\kappa_1 x^2 + \kappa_2 x + \kappa_3).$$

Then

$$h'(x) = 3\kappa_1 x^2 + 2\kappa_2 x + \kappa_3. \tag{H.4}$$

 $h(x) = \kappa_4$  has positive solutions is corresponding to that equation (H.3) has positive roots. If h'(x) = 0 has two positive roots  $B\eta_1$  and  $B\eta_2$  (assume  $B\eta_1 < B\eta_2$ ), and they satisfy

$$h(B\eta_2) < \kappa_4 < h(B\eta_1),$$

then (H.3) has three positive roots. From equation (H.4), if

$$\kappa_2^2 - 3\kappa_1\kappa_3 > 0,$$

then we can derive the solutions  $B\eta_1$  and  $B\eta_2$  of h'(x) = 0

$$B\eta_1 = \frac{-\kappa_2 - \sqrt{\kappa_2^2 - 3\kappa_1\kappa_3}}{3\kappa_1}$$
, and  $B\eta_2 = \frac{-\kappa_2 + \sqrt{\kappa_2^2 - 3\kappa_1\kappa_3}}{3\kappa_1}$ .

Since  $\kappa_1 > 0$  and  $\kappa_4 > 0$ ,  $B\eta_1$  is positive if and only if

 $\kappa_2 < 0$  and  $\kappa_3 > 0$ .

Let  $\kappa_2 = 0$  and  $\kappa_3 = 0$ , then

$$-\delta^{2} + \{4(\mathscr{R}_{0} - 1) + 6\theta_{c}\mathscr{R}_{0}\}\delta - (\theta_{c}\mathscr{R}_{0})^{2} = 0;$$

$$-\delta^2 + \{3(\mathscr{R}_0 - 1) + \theta_c \mathscr{R}_0\}\delta - \theta_c \mathscr{R}_0(\mathscr{R}_0 - 1) = 0.$$

The discriminants of  $\kappa_2 = 0$  and  $\kappa_3 = 0$  are given by

$$\Delta_2 = 16(\mathscr{R}_0 - 1)^2 + 48\theta_c \mathscr{R}_0(\mathscr{R}_0 - 1) + 32(\theta_c \mathscr{R}_0)^2 > 0;$$
  
$$\Delta_3 = 9(\mathscr{R}_0 - 1)^2 + 2\theta_c \mathscr{R}_0(\mathscr{R}_0 - 1) + (\theta_c \mathscr{R}_0)^2 > 0;$$

and  $\Delta_2 > \Delta_3$ . Define  $\delta_1$ ,  $\delta_2$  and  $\delta_3$ ,  $\delta_4$  are the solutions of  $\kappa_2 = 0$  and  $\kappa_3 = 0$ , respectively. Then

$$\delta_2 = \frac{1}{2} \{ 4(\mathscr{R}_0 - 1) + 6\theta_c \mathscr{R}_0 + \sqrt{\Delta_2} \};$$
  
$$\delta_4 = \frac{1}{2} \{ 3(\mathscr{R}_0 - 1) + \theta_c \mathscr{R}_0 + \sqrt{\Delta_3} \},$$

thus  $\delta_2 > \delta_4$ .

$$\kappa_2(\delta_3) = -\delta_3^2 + \{4(\mathscr{R}_0 - 1) + 6\theta_c \mathscr{R}_0\}\delta_3 - (\theta_c \mathscr{R}_0)^2,$$

where

$$-\delta_3^2 = \theta_c \mathscr{R}_0 (\mathscr{R}_0 - 1) - \{3(\mathscr{R}_0 - 1) + \theta_c \mathscr{R}_0\} \delta_3$$

Thus

$$\kappa_2(\delta_3) = \{ (\mathscr{R}_0 - 1) + 5\theta_c \mathscr{R}_0 \} \delta_3 - \theta_c \mathscr{R}_0 (\theta_c \mathscr{R}_0 - \mathscr{R}_0 + 1)$$
  
> 0.

Thus  $\delta_3 > \delta_1$ . Above all,  $\delta_2 > \delta_4 > \delta_3 > \delta_1$ . If  $\kappa_2 < 0$ , then  $\kappa_3 < 0$ . So if  $B\eta_1$  exists,  $B\eta_1$  can not be positive. So, if h'(x) = 0 has two positive roots,  $B\eta_1$  can not be positive. Thus, there is a unique solution for  $h(x) = \kappa_4$ .